# Math 409 Midterm 2 practice

**Name**: ____________________

**This exam has 4 questions, for a total of 100 points.**

Please answer each question in the space provided. No aids are permitted.

**Question 1. (40 pts)**

In each of the following eight cases, indicate whether the given statement is true or false. No justification is necessary.

(a) Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences in $\mathbb{R}$. Then the sequence $\{x_n y_n\}$ converges.

**Solution:** True.

(b) $\lim_{x\to 1} \frac{x^2-1}{x-1}$ does not exist.

**Solution:** False.

(c) Let $\{x_n\}$ be a sequence such that $x_n \in (0,1)$ for every $n \in \mathbb{N}$. Then $\{x_n\}$ has a subsequence which is Cauchy.

**Solution:** True. (This essentially follows from Bolzano-Weierstrass theorem.)

(d) $\lim_{x\to\infty} \frac{x - 2x^2 + 5x^3}{6 - x + x^2} = \infty$

**Solution:** True.

(e) Let $\{x_n\}$ be a sequence in $\mathbb{R}$ with the property that each of its subsequences has a convergent subsequence. Then $\{x_n\}$ is bounded.

**Solution:** True. (One can prove this by contradiction.)

(f) If a function is differentiable on $\mathbb{R}$, then it is uniformly continuous on $\mathbb{R}$.

**Solution:** False.

(g) Let $f$ be a function which is uniformly continuous on $\mathbb{R}$. Then the function $g$ defined by $g(x) = f(f(x))$ for all $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

**Solution:** True.

(h) If $f\colon (0, 1) \to \mathbb{R}$ is continuous and bounded, then $f$ is uniformly continuous.

**Solution:** False.

**Question 2. (20 pts)**

(a) Let $f$ be a function defined on an open interval containing a given point $a$. State what it means for $f(x)$ to converge to a number $L$ as $x$ approaches $a$.

> **Solution:** Omitted. You can find it in the textbook.

(b) Let $a \in \mathbb{R}$ and let $f$ and $g$ be functions on $\mathbb{R}$ such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Prove directly from the definition of a limit that $\lim_{x\to a}(f+g)(x)$ exists.

> **Solution:** By assumption, for $\forall \varepsilon > 0$, there exists $\delta_1 > 0$ such that
>
> $$|f(x) - f(a)| < \varepsilon/2$$
>
> for all $x$ with $0 < |x - a| < \delta_1$; similarly, there exists $\delta_2 > 0$ such that
>
> $$|g(x) - g(a)| < \varepsilon/2$$
>
> for all $x$ with $0 < |x - a| < \delta_2$.
> Let $\delta = \min\{\delta_1, \delta_2\}$. Then
>
> $$|(f+g)(x) - (f+g)(a)| < \varepsilon$$
>
> for all $x$ with $0 < |x - a| < \delta$.

**Question 3. (20 pts)**

(a) State the Extreme Value Theorem.

**Solution:** Omitted. You can find it in the textbook.

(b) Give an example of a function $f$ which is bounded on $[0,1]$ but does not have a maximum on $[0,1]$.

**Solution:** Define

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 0 & x = 1 \end{cases}$$

**Question 4. (20 pts)**

(a) State the Intermediate Value Theorem.

**Solution:** Omitted. You can find it in the textbook.

(b) Assuming the fact that the function $\cos x$ is continuous on $\mathbb{R}$, prove that there exists an $x \in \mathbb{R}$ such that $x^6 + x^4 + 1 = 2\cos x^3$.

**Solution:** Consider the function $g(x) = x^6 + x^4 + 1 - 2\cos x^3$. On the interval $[0, 1]$, we have

$$g(0) = -1 \text{ and } g(1) = 3 - 2\cos(1) > 0.$$

So we have $g(1) \leq 0 \leq g(0)$. It follows from the intermediate value theorem that there exists $x_0 \in [0, 1]$ such that $g(x_0) = 0$, that is, $x_0^6 + x_0^4 + 1 = 2\cos x_0^3$.